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RECONSTRUCTING TRAJECTORIES FROM MEASUREMENT DATA†

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Practical problems sometimes require the reconstruction of the continuous flight trajectory of an object from the results of coordinate measurements performed continuously or at discrete times, with errors. Here it is required to minimize the deviation of the reconstructed trajectory from the measured trajectory, also using available restrictions on the linear acceleration of the object. This paper presents a statement of the problem and a method of solving it. An example is given.

1. STATEMENT OF THE PROBLEM

Suppose that a material point is acted upon by a controlling acceleration $u(t)$ so that it performs one-dimensional motion described by the equations

$$\dot{x} = y, \quad \dot{y} = u \tag{1.1}$$

where x is the coordinate and y the velocity of the point. The acceleration $u(t)$ is constrained by the condition

$$|u(t)| \leqslant D \tag{1.2}$$

where D is a constant. Measurements of the coordinate $x(t)$ are performed during the interval $t \in [0, T]$, the results of the measurements being denoted by $h(t)$. It is required to reconstruct the trajectory of the object, i.e. to find a function $x(t)$ in the interval $t \in [0, T]$ so that constraint (1.2) is satisfied and the deviation $x(t) - h(t)$ is in some sense minimized.

The problem can be formalized as an optimal control problem in the following way.

It is required to determine the control $u(t)$ which minimizes the integral functional

$$J = \int_0^T \left[\frac{1}{2}\rho(t)(x - h(t))^2 + \frac{1}{2}Mu^2\right]dt \tag{1.3}$$

Here $\rho(t) \geqslant 0$ is a weight function, $M > 0$ is some initially undetermined constant, to be chosen later using the acceleration constraint (1.2). In (1.3) the function $x(t)$ is related to the control $u(t)$ by the differential equations (1.1). The value of the x coordinate at the initial and final instants $t = [0, T]$ are not assumed to be fixed, i.e. we have a problem with free end-points.

It is natural to specify the weight function $\rho(t)$ to be inversely proportional to the square of the mean-square error of the measurement of the coordinate $x(t)$. If the trajectory is reconstructed from discrete readings $h_k = h(t_k)$ at the instants t_k, $k = 0, \dots, N$, $t_0 = 0 < t_1 < \dots < t_N = T$, made with mean square errors σ_k, then the function $\rho(t)$ is given by the formula

$$\rho(t) = \rho_k\delta(t - t_k), \quad \rho_k = 1/\sigma_k^2 \tag{1.4}$$

where $\delta(\cdot)$ is the Dirac δ-function. We remark that to reconstruct a multidimensional trajectory the above problem must be solved independently for each spatial variable.

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2. THE OPTIMAL CONTROL PROBLEM

To determine the optimal trajectories $x(t)$ and accelerations $u(t)$ which minimize functional (1.3), we use Pontryagin's maximum principle. In our case the Hamiltonian function is

$$H = \psi_1 y + \psi_2 u + \tfrac{1}{2}\psi_0\left[\rho(t)(x-h(t))^2 + Mu^2\right] \tag{2.1}$$

From this we obtain equations for the adjoint variables

$$\dot\psi_1 = -\partial H/\partial x = -\psi_0\rho(t)(x-h(t)), \quad \dot\psi_2 = -\partial H/\partial y = -\psi_1, \quad \dot\psi_0 = 0 \tag{2.2}$$

Because both ends of the trajectory are free, the transversality conditions have the form

$$\psi_i(0) = \psi_i(T) = 0, \quad i = 1, 2, \quad \psi_0 \equiv -1 \tag{2.3}$$

When $\psi_0 = -1$ the function H defined by (2.2) reaches its maximum in u at the critical point, i.e. when $\partial H/\partial u = \psi_2 - Mu = 0$.

From this we obtain

$$u = \psi_2/M \tag{2.4}$$

Substituting (2.4) into system (1.1) and using (2.2), we find that the optimal trajectory is given by the set of equations

$$\dot x = y, \quad \dot y = \psi_2/M, \quad \dot\psi_2 = -\psi_1, \quad \dot\psi_1 = \rho(t)(x-h(t)) \tag{2.5}$$

with boundary conditions (2.3). After solving this boundary-value problem the optimal control is found from expression (2.4).

3. DISCRETE MEASUREMENTS

We will consider in more detail the discrete-measurement case which is important for practical applications. System (2.5) has an analytic solution when the function $\rho(t)$ is given by expression (1.4). Indeed, in the interval (t_k, t_{k+1}), $k = 0, \ldots, N-1$, i.e. between the instants when measurements are made, the right-hand side of the last equation in (2.5) vanishes, and system (2.5) is easily integrated

$$\begin{gathered} x = x_k^+ + y_k^+\tau + \frac{\psi_{2k}^+}{2M}\tau^2 - \frac{\psi_{1k}^+}{6M}\tau^3, \quad y = y_k^+ + \frac{\psi_{2k}^+}{M}\tau - \frac{\psi_{1k}^+}{2M}\tau^2 \\ \psi_2 = \psi_{2k}^+ - \frac{\psi_{1k}^+}{M}\tau, \quad \psi_1 = \psi_{1k}^+ \end{gathered} \tag{3.1}$$

Here $\tau = t - t_k$, and $x_k^+, y_k^+, \psi_{2k}^+, \psi_{1k}^+$ are the values of the corresponding variables at the instant $t = t_k + 0$ immediately after the kth measurement. From (3.1) we conclude that in the interval (t_k, t_{k+1}) the optimal trajectory $x(t)$ is the cubic spline.

The trajectory transition at the instant t_k is given by jump conditions which follow from (2.5) and (1.4)

$$x_k^+ = x_k, \quad y_k^+ = y_k, \quad \psi_{2k}^+ = \psi_{2k}, \quad \psi_{1k}^+ = \psi_{1k} + \rho_k(x_k - h_k) \tag{3.2}$$

Here $x_k, y_k, \psi_{1k}, \psi_{2k}$ are values of the corresponding variables at the instant $t = t_k - 0$ just before the kth measurement. Thus, at the instant of measurement, either the third time derivative of the coordinate or the first derivative of the control is discontinuous; the other variables remain continuous.

Combining (3.1) and (3.2), we obtain the transformation $\Phi_k: R^4 \to R^4$ governing the values of all variables at the instant $t = t_{k+1} - 0$ in terms of the values at the instant $t = t_k - 0$

$$x_{k+1} = x_k + y_k\tau_k + \frac{\psi_{2k}}{2M}\tau_k^2 - \frac{\psi_{1,k+1}}{6M}\tau_k^3, \quad y_{k+1} = y_k + \frac{\psi_{2k}}{M}\tau_k - \frac{\psi_{1,k+1}}{2M}\tau_k^2 \tag{3.3}$$

$$\psi_{2,k+1}=\psi_{2k}-\frac{\psi_{1,k+1}}{M}\tau_k,\quad \psi_{1,k+1}=\psi_{1k}+\rho_k(x_k-h_k)$$

Here and below $\tau_k = t_{k+1} - t_k$, $k = 0, \dots, N-1$. We rewrite (3.3) in matrix form

$$z_{k+1}=A_k z_k+b_k,\quad k=0,\dots,N-1 \tag{3.4}$$

Here $z_s = (x_s, y_s, \psi_{2s}, \psi_{1s})^T$, $s = 0, \dots, N$, the symbol T denotes transposition and we have introduced the notation

$$A_k=\begin{Vmatrix} 1-\frac{\tau_k^3}{6M}\rho_k & \tau_k & \frac{\tau_k^2}{2M} & \frac{\tau_k^3}{6M} \\ -\frac{\tau_k^2}{2M}\rho_k & 1 & \frac{\tau_k}{M} & \frac{\tau_k^2}{2M} \\ -\tau_k\rho_k & 0 & 1 & -\tau_k \\ \rho_k & 0 & 0 & 1 \end{Vmatrix},\quad b_k=\begin{Vmatrix} \frac{\tau_k^3}{6M} \\ \frac{\tau_k^2}{2M} \\ \tau_k \\ -1 \end{Vmatrix}\rho_k h_k \tag{3.5}$$

Similarly, we introduce the mapping $\Phi_k^\delta: R^4 \to R^4$ corresponding to the transformation (3.2) and write it in matrix form

$$z_k^+=A_k^\delta z_k+b_k^\delta,\quad k=0,\dots,N$$

$$A_k^\delta=\begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \rho_k & 0 & 0 & 1 \end{Vmatrix},\quad b_k^\delta=\begin{Vmatrix} 0 \\ 0 \\ 0 \\ -\rho_k h_k \end{Vmatrix} \tag{3.6}$$

In order to find the optimal trajectory and control it is sufficient to determine the values of all components of the vector z_0 at the instant $t_0 - 0$. The values of the coordinate and control at all instants $t = t_k - 0$ and $t = t_k + 0$, $k = 0, \dots, N$ are then reproduced using mappings (3.4) and (3.6), while the values at the intermediate instants $t \in (t_k, t_{k+1})$ are determined using formulae (3.1).

It would appear that the value of z_0 can be determined by the following simple method. Find the map

$$\Phi=\Phi_N^\delta\circ\Phi_{N-1}\circ\dots\circ\Phi_0 \tag{3.7}$$

turning the initial point z_0 into the final point z_N. To determine variables satisfying boundary conditions (2.3) it is necessary to solve the system of linear equations

$$z_N^+=\Phi(z_0) \tag{3.8}$$

Here two components for each of the vectors z_N^+ and z_0 are known: $\psi_{N1}^+ = \psi_{N2}^+ = \psi_{01} = \psi_{02} = 0$. Thus the problem of finding z_0 reduces to the algebraic operation of matrix multiplication in the calculation of the explicit form of the map (3.7) and to subsequent solution of the linear system of equations (3.8).

However, this simplest method fails because of fundamental numerical difficulties.

The problem is that for all k the matrix A_k from (3.5) has two eigenvalues with moduli greater than unity. The eigenvalues λ of the matrix A_k are governed by the characteristic equation

$$P(\lambda)=\lambda^4+(\kappa/6-4)\lambda^3+(2\kappa/3+6)\lambda^2+(\kappa/6-4)\lambda+1=0,\quad \kappa=\rho_k\tau_k^3/M>0 \tag{3.9}$$

This equation is reciprocal, and its roots are positioned symmetrically with respect to the unit circle. They would only lie on the unit circle itself when $\kappa = 0$, that is impossible (see (3.9)). Hence there are always two roots with modulus greater than unity. The presence of these eigenvalues leads to instability and to an exponential increase in the matrix coefficients obtained when computing mapping (3.7), and an exponential increase in the numerical errors also. All this makes calculation impossible even for fairly small N. We must therefore abandon the calculation of mapping (3.7) and use another method.

4. MATRIX PIVOTAL CONDENSATION

We shall use the matrix pivotal condensation method [1]. We rewrite the matrices and vectors that occur in transformation (3.4) in the following partitioned form, together with the transformation itself

$$z_k = \left\| \begin{matrix} \xi_{1k} \\ \xi_{2k} \end{matrix} \right\|, \quad \xi_{1k} = \left\| \begin{matrix} x_k \\ y_k \end{matrix} \right\|, \quad \xi_{2k} = \left\| \begin{matrix} \psi_{2k} \\ \psi_{1k} \end{matrix} \right\|, \quad A_k = \left\| \begin{matrix} A_{11}^k & A_{12}^k \\ A_{21}^k & A_{22}^k \end{matrix} \right\|$$

$$b_k = \left\| \begin{matrix} b_{1k} \\ b_{2k} \end{matrix} \right\|, \quad k = 0, \ldots, N \tag{4.1}$$

$$\xi_{1,k+1} = A_{11}^k \xi_{1k} + A_{12}^k \xi_{2k} + b_{1k}, \quad \xi_{2,k+1} = A_{21}^k \xi_{1k} + A_{22}^k \xi_{2k} + b_{2k}$$

Here A_{ij} and b_{ik} are (2×2)-matrices and 2-vectors, corresponding to vectors ξ_{1k} and ξ_{2k}. We make the substitution

$$\xi_{2k} = Q_k \xi_{1k} + q_k, \quad k = 0, \ldots, N \tag{4.2}$$

in (4.1), where Q_k and q_k are an initially unknown (2×2)-matrix and 2-vector. We obtain

$$\begin{aligned} &\xi_{1,k+1} = \left(A_{11}^k + A_{12}^k Q_k\right)\xi_{1k} + A_{12}^k q_k + b_{1k} \\ &Q_{k+1}\xi_{1,k+1} + q_{k+1} = \left(A_{21}^k + A_{22}^k Q_k\right)\xi_{1k} + A_{22}^k q_k + b_{2k} \end{aligned} \tag{4.3}$$

We now substitute $\xi_{1,k+1}$ from the first equation in (4.3) into the second, and equating the coefficients of the ξ_{1k} and the free terms on the left- and right-hand sides, we obtain the system

$$\begin{aligned} &Q_{k+1}\left(A_{11}^k + A_{12}^k Q_k\right) = A_{21}^k + A_{22}^k Q_k \\ &Q_{k+1}\left(A_{12}^k q_k + b_{1k}\right) + q_{k+1} = A_{22} q_k + b_{2k}, \quad k = 0, \ldots, N-1 \end{aligned} \tag{4.4}$$

Note that if the matrices Q_k, Q_{k+1} and vectors q_k, q_{k+1} satisfy system (4.4), the equation obtained from (4.3) after eliminating $\xi_{1,k+1}$ is satisfied identically by ξ_{1k}. Relations (4.4) are recurrence relations connecting the Q_k, q_k with the Q_{k+1}, q_{k+1}.

We now choose values for Q_N, q_N so that boundary condition (2.3) at the right end of the trajectory is satisfied identically by any ξ_{1N}. We recall that according to (4.1) the values ξ_{1N}, ξ_{2N} correspond to values of the coordinates and adjoint variables at the instant $t_N - 0$. Hence, in order to satisfy the boundary condition $\xi_2(t_N + 0) \equiv 0$, it is necessary to apply transformation (3.6) to the vectors ξ_{1N}, ξ_{2N}. We obtain

$$\xi_2(t_N + 0) = \left\| \begin{matrix} 0 & 0 \\ \rho_N & 0 \end{matrix} \right\| \xi_{1N} + \xi_{2N} - \left\| \begin{matrix} 0 \\ \rho_N h_N \end{matrix} \right\| \equiv 0 \tag{4.5}$$

We substitute the expression for ξ_{2N}, given by the substitution (4.2), into (4.5), and in the resulting relation we equate the coefficients of ξ_{1N} and the free terms to zero. We obtain

$$Q_N = \left\| \begin{matrix} 0 & 0 \\ -\rho_N & 0 \end{matrix} \right\|, \quad q_N = \left\| \begin{matrix} 0 \\ \rho_N h_N \end{matrix} \right\| \tag{4.6}$$

From relations (4.4) we obtain recurrence formulae expressing Q_k, q_k in terms of Q_{k+1}, q_{k+1}

$$Q_k = \left(A_{22}^k - Q_{k+1}A_{12}^k\right)^{-1}\left(Q_{k+1}A_{11}^k - A_{21}^k\right), \quad q_k = \left(A_{22}^k - Q_{k+1}A_{12}^k\right)^{-1}\left(Q_{k+1}b_{1k} - b_{2k} + q_{k+1}\right) \tag{4.7}$$

The first equation in (4.7) gives the non-linear relation between the matrices Q_k and Q_{k+1}, while the second gives the linear relation between the vectors q_k and q_{k+1}.

Using (4.7) and initial conditions (4.6) we sequentially obtain the matrices $Q_{N-1}, \ldots, Q_0$ and vectors $q_{N-1}, \ldots, q_0$. Having computed Q_0 and q_0, and then, using substitution (4.2) and boundary condition (2.3) at the initial instant ($\xi_{20} \equiv 0$), we obtain the initial point of the optimal trajectory

$$\left\| \begin{matrix} x_0 \\ y_0 \end{matrix} \right\| = \xi_{10} = -Q_0^{-1} q_0 \tag{4.8}$$

The initial value $z_0 = (\xi_{10}\xi_{20})^T$ is thus determined by relation (4.8). The optimal trajectory is now easily reconstructed at the nodes t_k using expressions (4.2) and the first expression in (4.3), and at the intermediate points using (3.10).

For a final solution of the problem it remains to choose the parameter M so that the acceleration condition (1.2) is satisfied. The maximum acceleration $u_{\max}$ along the optimal trajectory is a function of M, and $u_{\max} \to \infty$ as $M \to 0$ and $u_{\max} \to 0$ as $M \to \infty$. Using a trial method it is easy to find numerically the value of M for which $|u_{\max}| = D$.

Thus the solution of the problem of reconstructing the trajectory in terms of measurement data reduces to calculating the matrices Q_k and vectors q_k by recurrence formulae (4.7) followed by a calculation of the trajectories from formulae (4.2), (4.3) and (3.10), together with the selection of the parameter M. We note that the most onerous part of this procedure—the calculation of the matrices Q_k using the first (non-linear) relation in (4.7)—does not depend on the results of measurements of h_k occurring in the vectors b_k in accordance with (3.6). Hence the matrices Q_k can be calculated in advance of the measurements. When different sets of measurements are performed with the same errors, there is no need to repeat the calculation of the Q_k, which is performed only once.

The algorithm described for solving the trajectory reconstruction problem was implemented on a computer. Calculations showed the effectiveness and high accuracy of the method.

5. STABILITY OF THE PIVOTAL CONDENSATION METHOD

The pivotal condensation method described in Section 4 is similar to the matrix pivotal condensation method described in [1, pp. 106–111]. However, a check showed that the sufficient conditions for the well-posedness and stability of the algorithm that were obtained in [1] are not satisfied in our case. We will show that those sufficient conditions can be widened considerably in cases when the A_k matrices do not depend on the number k, i.e. when $A_k = A$, where A is a constant matrix.

We shall say that the pivotal condensation method is well-posed for given end conditions Q_N, q_N in reverse time, if for all $k = N-1, \dots, 0$ the matrices Q_k and vectors q_k are uniquely defined. From formulae (4.7) it obviously follows that the condition

$$\det(A_{22} - Q_{k+1}A_{12}) \neq 0, \quad k = N-1, \dots, 0 \tag{5.1}$$

is necessary and sufficient for the well-posedness Below we shall omit the superscript k in the matrices A_{ij}^k.

We shall say that the pivotal condensation method is asymptotically stable for end condition Q_N in reverse time if the first of the mappings (4.7) has an asymptotically stable fixed point $Q_{k+1} = Q_k = Q_*$ and the matrix Q_N lies in the domain of attraction of this point, and the second mapping in (4.7) has an asymptotically stable fixed point $q_{k+1} = q_k = 0$ if b_{1k}, b_{2k} are set equal to zero. Moreover, we shall require the first mapping in (4.3) to have an asymptotically stable fixed point $\xi_{1,k+1} = \xi_{1k} = 0$ if q_k, b_{1k} are set equal to zero.

We note that the first two conditions guarantee the computational stability of the determination of the condensational sequences of matrices Q_k and vectors q_k, and the last conditions guarantee the stability of the reconstruction of the trajectory.

In a similar way one can define the well-posedness and asymptotic stability of the pivotal condensation method in forward time.

Before obtaining sufficient conditions for the well-posedness and asymptotic stability of the pivotal condensation method we will first derive transformation formulae for the matrices Q_k and vectors q_k for a linear change of variables in phase space. We consider the mapping

$$\Phi_k(z) = Az + b_k : R^4 \to R^4, \quad k = 0, \dots, N-1 \tag{5.2}$$

and the associated mapping of the pivotal condensation method (4.4). We make the linear change of variables

$$z = Cz', \quad \Phi_k = C\Phi_k' \tag{5.3}$$

defined by the non-degenerate matrix C. Here and below, variables without a prime are old, and primed variables are new. We rewrite (5.3) in partitioned form

$$\begin{aligned}\xi_1 &= C_{11}\xi_1' + C_{12}\xi_2', \\ \xi_2 &= C_{21}\xi_1' + C_{22}\xi_2'\end{aligned} \quad C = \begin{Vmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{Vmatrix}, \quad z = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \tag{5.4}$$

With this change of variables the mapping (5.2) becomes

$$\Phi_k'(z') = A'z' + b_k, \quad A' = C^{-1}AC, \quad b_k' = C^{-1}b_k, \quad k = 0, \ldots, N-1 \tag{5.5}$$

We now derive transformation formulae for Q_k, q_k. We shall require them to satisfy the relations

$$\xi_{2k}' = Q_k'\xi_{1k}' + q_k', \quad k = 0, \ldots, N \tag{5.6}$$

which are similar to (4.2). Substituting (5.6) into (5.4), and then into the expressions obtained for ξ_{1k} and ξ_{2k} in (4.2), we have

$$Q_k(C_{11} + C_{12}Q_k')\xi_{1k}' + Q_kC_{12}q_k' + q_k = (C_{21} + C_{22}Q_k')\xi_{1k}' + C_{22}q_k' \tag{5.7}$$

Further, since the Q_k', q_k' do not depend on the vector ξ_{1k}' (in the same way that Q_k, q_k do not depend on ξ_{1k}), we equate the coefficients of ξ_{1k}' and the free terms on the left- and right-hand sides of (5.7). As a result we obtain the required transition formulae

$$Q_k(C_{11} + C_{12}Q_k') - (C_{21} + C_{22}Q_k') = 0, \quad q_k = (C_{22} - Q_kC_{12})q_k' \tag{5.8}$$

As well as (5.8), we obtain formulae connecting Q_{k+1}', q_{k+1}' with Q_k', q_k', derived in the same way as (4.7) for the old variables

$$\begin{aligned} Q_k' &= (A_{22}' - Q_{k+1}'A_{12}')^{-1}(Q_{k+1}'A_{11}' - A_{21}') \\ q_k' &= (A_{22}' - Q_{k+1}'A_{12}')^{-1}(Q_{k+1}'b_{1k} - b_{2k} + q_{k+1}) \end{aligned} \tag{5.9}$$

We now assume that the change of variables (5.3) reduces matrix A to Jordan form.

In the general situation the eigenvalues $\lambda_1, \ldots, \lambda_4$ of the matrix A are different and the matrix A' is diagonal. Hence $A_{12}' = A_{21}' = 0$. Equations (5.9) can therefore be rewritten in the form

$$Q_k' = (A_{22}')^{-1}Q_{k+1}'A_{11}', \quad q_k' = (A_{22}')^{-1}(Q_{k+1}'b_{1k}' - b_{2k}' + q_{k+1}') \tag{5.10}$$

To fix our ideas we take

$$A_{11}' = \begin{Vmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{Vmatrix}, \quad A_{22}' = \begin{Vmatrix} \lambda_3 & 0 \\ 0 & \lambda_4 \end{Vmatrix}$$

The relation between the matrices Q_k' and Q_{k+1}' is now linear, unlike relations (4.7). It is obvious that the pivotal condensation method in the new variables is always well-posed for any starting conditions $Q_N'q_N'$ and any number of iterations N if $\lambda_3 \neq 0$, $\lambda_4 \neq 0$. Moreover, the mapping determined by the first formula in (5.10) has a unique fixed point $Q_* = 0$ which is globally asymptotically stable when $\max(|\lambda_1|, |\lambda_2|) < \min(|\lambda_3|, |\lambda_4|)$ and unstable if the opposite inequality is satisfied. The map determined by the second formula in (5.10) has, if one sets b_{1k}' and b_{2k}' equal to zero, a fixed point $q_* = 0$ which is globally asymptotically stable when $\min(|\lambda_3|, |\lambda_4|) > 1$. The reconstructed trajectory is then generated by formula (5.6) together with

$$\xi_{1,k+1}' = A_{11}'\xi_{1k}' + b_{1k}' \tag{5.11}$$

which is obtained in the same way as the first formula in (4.3). If one equates b_{1k}' to zero, mapping (5.11) has a fixed point $\xi_* = 0$ which is globally asymptotically stable when $\max(|\lambda_1|, |\lambda_2|) < 1$.

The following theorem has therefore been proved.

Theorem 1. Suppose that the eigenvalues of the matrix A defining the mapping (5.2) are distinct, and two of them lie inside, and two outside, the unit circle. Then, in variables in which the matrix A has diagonal form, and its eigenvalues are arranged in order of increasing modulus ($|\lambda_1| \leqslant |\lambda_2| < 1 <$

$|\lambda_3| \leq |\lambda_4|$), the pivotal condensation method is well-posed and asymptotically stable in reverse time for all initial conditions.

We note that from formulae (5.10)–(5.11) one can similarly obtain sufficient conditions for well-posedness and asymptotic stability of the pivotal condensation method in direct time, in the form $\min(|\lambda_1|, |\lambda_2|) > 1 > \max(|\lambda_3|, |\lambda_4|)$.

Theorem 2. Let the conditions of Theorem 1 be satisfied. If the starting value of the matrix Q_N is such that

$$\det\left\|C_{22} - Q_N C_{22}\right\| \neq 0, \quad \det\left\|C_{11} + C_{12}Q_0'\right\| \neq 0, \quad \det\left\|C_{21} + C_{22}Q_0'\right\| \neq 0 \tag{5.12}$$

then for an arbitrary value of q_N the pivotal condensation method is well-posed and asymptotically stable in reverse time for the original variables.

Proof. For specified values of Q_N and q_n the sequences Q_k and q_k can be constructed not only from formulae (4.7), but also by means of the change of variables (5.3) by formulae (5.8) and (5.10), as shown in the diagram

$$\begin{array}{ccc} & Q_N, q_N \xrightarrow{(4.7)} Q_{N-1}, q_{N-1} \xrightarrow{(4.7)} \cdots \to Q_0, q_0 & \\ (5.8)\downarrow & & \uparrow(5.8) \\ & Q_N', q_N' \xrightarrow[(5.10)]{} Q_{N-1}', q_{N-1}' \xrightarrow[(5.10)]{} \cdots \to Q_0', q_0' & \end{array}$$

From relation (5.8) with $k = N$ and $k = 0$ we obtain the following conditions of solvability. The transition from Q_N, q_N to Q_N', q_N' is defined in $\det\|C_{22} - Q_N C_{12}\| \neq 0$, while the transition from Q_0', q_0' to Q_0, q_0 is defined if $\det\|C_{11} - C_{12}Q_0'\| \neq 0$. Because the matrix Q_0^{-1} is necessary to determine the initial conditions x_0, y_0, Q_0 must be non-degenerate. It follows from (5.8) that the necessary and sufficient condition for this is $\det\|C_{11} + C_{12}Q_0'\| \neq 0$, $\det\|C_{21} + C_{22}Q_0'\| \neq 0$. Theorem 2 is proved.

We will now apply the results obtained to the matrix A defined by expression (3.5) with $\tau_k = \tau$, $\rho_k = \rho$. Because the characteristic equation (3.9) of A is reciprocal, the eigenvalues of this matrix are distinct and arranged symmetrically with respect to the unit circle. Hence the conditions of Theorem 1 are satisfied. By Theorem 2 the pivotal condensation method is well-posed and asymptotically stable if inequalities (5.12) are satisfied for finite values of Q_N, q_N from (4.6). The validity of these inequalities can always be established numerically for chosen values of the system parameters τ, ρ M. We note that the matrix Q_0' can be assumed to be zero after a relatively small number of iterations N, because its coefficients are of order $O(\varepsilon^N)$ where $\varepsilon = |\lambda_2|/|\lambda_3| < 1$. In particular, for the example considered below we have $N = 40$, $\varepsilon < 1/2$. Hence we can put $Q_0' = 0$ in (5.12) with a high degree of accuracy, and then verify the conditions

$$\det\left\|C_{22} - Q_N C_{12}\right\| \neq 0, \quad \det\left\|C_{11}\right\| \neq 0, \quad \det\left\|C_{21}\right\| \neq 0$$

This verification was performed numerically in the example to be considered.

6. EXAMPLE

We shall consider a model problem for the motion of a helicopter, which from an initial state of horizontal flight performs a manoeuvre to increase its height. We assume that the horizontal component of its velocity is constant both in magnitude and direction. The vertical motion is the result of a controlling acceleration. Measurements of the height h are made at intervals of $\tau = 1$ s. The manoeuvre takes 15 s and the total time of the motion is 25 s. Thus $N = 25$. The quantity σ is taken to be 2 m. The result of smoothing the trajectory, according to the algorithm described above, when the absolute magnitude of the controlling acceleration does not exceed $D = 2$ m/s^2 (see (1.2)) is shown in Fig. 1. Time is shown horizontally in seconds and the height vertically in metres. The small circles show the measured values, and the continuous curves shows the smoothed trajectory. To illustrate the dependence of the degree of smoothness of the trajectory on the magnitude of the permitted acceleration, the same original trajectory was smoothed with a maximum permissible acceleration of 30 m/s^2. The result is shown by the dashed line.

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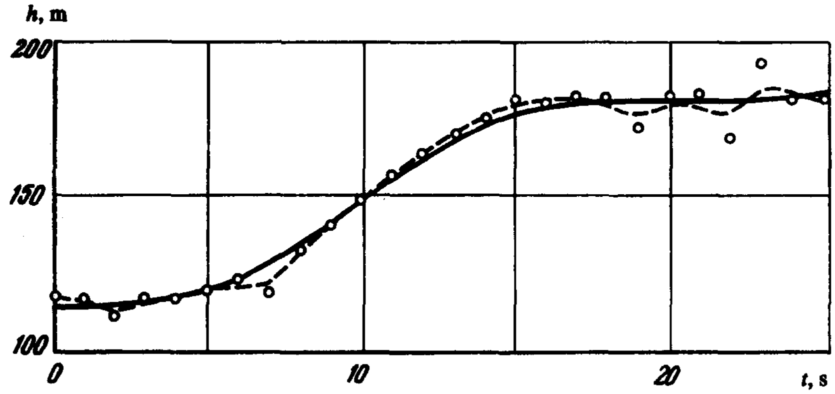

Fig. 1.

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